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# Magneto-optical conductivity in graphene

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## Abstract

Landau level quantization in graphene reflects the Dirac nature of its quasiparticles and has been found to exhibit an unusual integer quantum Hall effect. In particular, the lowest Landau level can be thought of as shared equally by electrons and holes, and this leads to characteristic behaviour of the magneto-optical conductivity as a function of frequency  $\Omega$  for various values of the chemical potential  $\mu$ . Particular attention is paid to the optical spectral weight under various absorption peaks and its redistribution as  $\mu$  is varied. We also provide results for magnetic field  $B$  as well as chemical potential sweeps at selected fixed frequencies, which can be particularly useful for possible measurements in graphene. Both diagonal and Hall conductivities are considered.

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Recent experimental studies [1, 2] of the dynamics of electrons and holes in graphene (a single atomic layer of graphite [3]) have revealed unusual behaviour related to the Dirac nature of its quasiparticles. Two dimensional graphene has a honeycomb lattice structure with two atoms per unit cell. Its band structure consists of two inequivalent pairs of cones with apices at the Brillouin zone corners. For zero chemical potential the lower energy cones are completely filled and the upper empty. In a graphene device an applied gate voltage can be used to introduce electrons in the upper band or, by voltage reversal, holes in the lower band (cones).

The quasiparticles in graphene obey the Dirac [4, 5] rather than the Schrödinger equation and this has profound implications for their dynamics. The unconventional quantum Hall effect was expected theoretically [6–10] and recently observed [1, 2] to have half integer (divided by spin and valley degeneracy) rather than integer filling factors. The predicted phase shift of

$\pi$  in the de Haas–van Alphen [11, 12]<sup>3</sup> and Shubnikov–de Haas [14] oscillations was also seen experimentally [1, 2]. Another feature related to the Dirac-like character of the carriers in graphene seen in the dc measurements is a finite effective cyclotron mass for the massless Dirac quasiparticles, which varies as the square root of the number of carriers [11, 14, 1, 2].

In this paper we consider the magneto-optical conductivity of graphene. Work without a magnetic field includes the calculations of Ando *et al* [15], who considered the effect on the frequency dependent conductivity of short- and long-range scatterers in a self-consistent Born approximation. More recent work [16] describes several anomalous properties of the microwave conductivity of graphene with, as well as without, magnetic field. These properties are directly related to the Dirac nature of the quasiparticles. Several analytic formulae for the longitudinal as well as Hall ac conductivity are given in the paper [8]. They also present extensive results for dc properties and preliminary data on the real part of  $\sigma_{xx}(\Omega)$  versus  $\Omega$  in the optical region. Another extensive work by Peres *et al* [9] on the ac conductivity in graphene treats localized impurities in a self-consistent fashion as well as extended edge and grain boundaries including also effects of electron–electron interactions and self-doping.

In this paper we follow most closely [8], which we extend in several directions. An aim is to provide simpler analytic formulae, which should prove useful in the analysis of experiment and check on their accuracy. Another is to consider magnetic field as well as chemical potential sweeps possible in graphene field effect transistor devices [1–3, 17]. Recent work by Li *et al* in organic metals [18] has demonstrated that the ac measurements are also possible in such devices. Consistent with our aim, we consider impurities only in the simplified scattering rate model and neglect real part renormalizations, although these could easily be included if wished. While the renormalization effects beyond those included in this simplified model might become important for the interpretation of future experiments, we point out that so far the free quasiparticle model with associated transport lifetime has been remarkably successful in understanding the dc results of [1, 2].

In section 2 we relate the magneto-optical conductivity tensor to the Dirac fermionic Green’s function through the Kubo formula. The general formulae obtained can be greatly simplified and closed form expressions are obtained in two cases. In general the fermionic self-energy can depend on energy and Landau level index as well as the temperature and value of the external magnetic field  $B$ . Under the assumption that variations with Landau level index  $n$  can be neglected, the sum over transitions between neighbouring Landau levels can be carried out explicitly and a closed form expression is obtained for conductivity in terms of the digamma function. A single integral over an internal frequency remains. The expression obtained is suitable for calculations of the ac conductivity for any value of temperature, chemical potential and magnetic field. Its microwave frequency limit has been used to describe properties of graphene in [16]. In the case when energy dependence of the fermionic self-energy can be neglected, the internal integration over energy can be done and what remains is the sum over the Landau level index  $n$  of Lorentzian forms multiplied by thermal factors and algebraic weighting factors. The weighting factors depend on the Landau level energies as well as the excitonic gap (see e.g. [19–21]), should one wish to include this possibility. When we compare numerical results obtained from the Lorentzian model and from the previous more general expressions in the limit of constant scattering rate, we find good quantitative agreement between the two. This provides support for the analysis of experimental data [22]. In section 2.3 we consider the low field limit for the Lorentzian model derived in section 2.2 and establish its correspondence with previously known results. In section 3.1 we present the numerical results for the real part

<sup>3</sup> The fact that the phase shift of  $\pi$  in graphite is related to nonzero Berry’s phase was discussed by Mikitik and Sharlai [13].

of the diagonal conductivity as a function of photon energy  $\Omega$  for fixed value of magnetic field and various values of chemical potential. We also provide results for fixed photon energy while sweeping either chemical potential or external magnetic field, which should prove useful in comparing with experiments. The effect of opening of an excitonic gap on the absorption lines is described. In section 3.2 a discussion of optical spectral weight redistribution by the magnetic field is given. Section 3.3 is structured in parallel to section 3.1 but deals with the absorptive part of the off-diagonal Hall magneto-optical conductivity. A discussion and conclusions are given in section 4. Some of the algebra needed in this work is found in the appendix.

## 2. Analytic expressions for optical conductivity

The optical conductivity tensor is calculated using the Kubo formula

$$\sigma_{ij}(\Omega) = \frac{\Pi_{ij}^R(\Omega + i0)}{i\Omega}, \quad (1)$$

where  $\Pi_{ij}^R(\Omega)$  is the retarded current–current correlation function, which in the bubble approximation is given by

$$\begin{aligned} \Pi_{ij}(\Omega + i0) &= e^2 v_F^2 \int_{-\infty}^{\infty} d\omega d\omega' \frac{n_F(\omega') - n_F(\omega)}{\omega - \omega' - \Omega - i0} \\ &\times \int \frac{d^2k}{(2\pi)^2} \text{tr} [\gamma^i A(\omega, \mathbf{k}) \gamma^j A(\omega', \mathbf{k})], \quad i = 1, 2 \end{aligned} \quad (2)$$

where  $n_F(\omega)$  is the Fermi distribution function  $n_F(\omega) = 1/[\exp((\omega - \mu)/T) + 1]$ ;  $\text{tr}$  not only takes care of the  $4 \times 4$   $\gamma$  matrices [ $\gamma^v = \sigma_3 \otimes (\sigma_3, i\sigma_2, -i\sigma_1)$ ], but also includes the summation over flavour (spin) index. Here

$$\begin{aligned} A(\omega, \mathbf{k}) &= e^{-\frac{\mathbf{k}^2}{|eB|}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma_n(\omega)}{2\pi M_n} \left[ \frac{(\gamma^0 M_n + \Delta) f_1(\mathbf{k}) + f_2(\mathbf{k})}{(\omega - M_n)^2 + \Gamma_n^2(\omega)} \right. \\ &\quad \left. + \frac{(\gamma^0 M_n - \Delta) f_1(\mathbf{k}) - f_2(\mathbf{k})}{(\omega + M_n)^2 + \Gamma_n^2(\omega)} \right] \end{aligned} \quad (3)$$

is the spectral function (decomposed over Landau levels) associated with the translation invariant part of the Dirac fermion Green's function in an external magnetic field  $\mathbf{B}$  applied perpendicular to the plane along the positive  $z$  direction (see e.g. [21, 11]). In equation (3)

$$f_1(\mathbf{k}) = 2 \left[ P_- L_n \left( \frac{2c\mathbf{k}^2}{|eB|} \right) - P_+ L_{n-1} \left( \frac{2c\mathbf{k}^2}{|eB|} \right) \right], \quad f_2(\mathbf{k}) = 4v_F \mathbf{k} \boldsymbol{\gamma} L_{n-1}^1 \left( \frac{2c\mathbf{k}^2}{|eB|} \right) \quad (4)$$

with  $P_{\pm} = (1 \pm i\gamma^1 \gamma^2 \text{sgn}(eB))/2$  being projectors and  $L_n^\alpha(z)$  the generalized Laguerre polynomials. By definition,  $L_n(z) \equiv L_n^0(z)$  and  $L_{-1}^\alpha(z) \equiv 0$ . The energies of the relativistic Landau levels in equation (3) are

$$E_n = \pm M_n, \quad M_n = \sqrt{\Delta^2 + 2n v_F^2 |eB|/c}, \quad (5)$$

where the energy scale associated with the magnetic field expressed in the units of temperature reads

$$\frac{eB v_F^2}{c} \rightarrow \frac{eB \hbar v_F^2}{c} \frac{1}{k_B^2} = 8.85 \times 10^{-8} \text{ K}^2 v_F^2 (\text{m s}^{-1}) B (\text{T}), \quad (6)$$

where  $v_F$  is the Fermi velocity in graphene given in  $\text{m s}^{-1}$  and the field  $B$  is given in tesla. In the following we set  $\hbar = k_B = 1$ , and in some places  $e = c = 1$ , unless stated explicitly otherwise.

For the numerical calculations we use the value  $v_F \approx 10^6 \text{ m s}^{-1}$  [1, 2], which leads to the relationship  $eB \rightarrow (8.85 \times 10^4 \text{ K}^2) B \text{ (T)}$ . We consider relatively low fields  $B \lesssim 17 \text{ T}$ , where spin splitting is unresolved [23], so that we can assume that the above mentioned summation over flavour index simply gives  $N_f = 2$  in all expressions below. The Landau level energies (5) also include an excitonic gap  $\Delta$ . The physical meaning of this singlet excitonic gap is directly related to the electron density imbalance between the A and B sublattices of the bi-particle hexagonal lattice of graphene [19, 20] and there are strong indications [24] that it was indeed observed in recent experiments [23, 25]. We will see here that optical measurements made on graphene can be very useful in investigations of the excitonic gap.

Finally, the scattering rate  $\Gamma_n(\omega)$  is expressed via the retarded fermion self-energy,  $\Gamma_n(\omega) = -\text{Im}\Sigma_n^R(\omega)$ , which in general depends on the energy, temperature, field and the Landau level index  $n$ . This self-energy, which in general also has a real part, has to be determined self-consistently from the Schwinger–Dyson equation. This equation can be solved analytically [20] and numerically as was done in [6, 9]. In our paper we consider  $\Gamma_n(\omega)$  as a phenomenological parameter for two cases: (i)  $\Gamma(\omega) = \Gamma_n(\omega)$  is independent of the Landau level index  $n$  and (ii)  $\Gamma_n = \Gamma_n(\omega)$  is independent of the energy  $\omega$ . Under these assumptions the optical conductivity can be studied analytically.

### 2.1. Frequency dependent scattering rate

Assuming that  $\Gamma_n(\omega)$  is independent of the Landau level index, i.e.  $\Gamma(\omega) = \Gamma_n(\omega)$ , one can calculate the sum over Landau levels and express the diagonal ac conductivity in the closed form [8]

$$\begin{aligned} \text{Re } \sigma_{xx}(\Omega) = & \frac{e^2 N_f}{4\pi^2 \Omega} \int_{-\infty}^{\infty} d\omega [n_F(\omega) - n_F(\omega')] \text{Re} \left\{ \frac{2B}{\Delta^2 - (\tilde{\omega} + i\Gamma)^2} [\Xi_1(-B) - \Xi_2(-B)] \right. \\ & + [\Xi_1(-B) + \Xi_1(+B) - \Xi_2(-B) - \Xi_2(+B)] \\ & \left. \times \psi \left( \frac{\Delta^2 - (\tilde{\omega} + i\Gamma)^2}{2B} \right) + (\tilde{\omega} \leftrightarrow \tilde{\omega}', \Gamma \leftrightarrow \Gamma') \right\}. \end{aligned} \quad (7)$$

Here  $\psi$  is the digamma function, we denoted  $B \equiv v_F^2 |eB|/c$ , and also included the renormalization of energy caused by the real part of self-energy  $\tilde{\omega}(\omega) = \omega - \text{Re } \Sigma(\tilde{\omega})$ ,  $\tilde{\omega}' = \tilde{\omega}(\omega + \Omega)$ ,  $\Gamma = \Gamma(\tilde{\omega})$ ,  $\Gamma' = \Gamma(\tilde{\omega}')$  and introduced the following short-hand notations:

$$\begin{aligned} \Xi_1(\pm B) & \equiv \Xi_1(\tilde{\omega}, \tilde{\omega}', \Gamma, \Gamma', \pm B) = \frac{(\tilde{\omega}' + i\Gamma')(\tilde{\omega} + i\Gamma) - \Delta^2}{[\tilde{\omega} - \tilde{\omega}' + i(\Gamma - \Gamma')][\tilde{\omega} + \tilde{\omega}' + i(\Gamma + \Gamma')] \pm 2B}, \\ \Xi_2(\pm B) & \equiv \Xi_2(\tilde{\omega}, \tilde{\omega}', \Gamma, \Gamma', \pm B) = \frac{(\tilde{\omega}' - i\Gamma')(\tilde{\omega} + i\Gamma) - \Delta^2}{[\tilde{\omega} - \tilde{\omega}' + i(\Gamma + \Gamma')][\tilde{\omega} + \tilde{\omega}' + i(\Gamma - \Gamma')] \pm 2B}. \end{aligned} \quad (8)$$

The advantage of equation (7) is that the  $\psi$  function contains the contribution to conductivity from *all* transitions between neighbouring Landau levels. An infinite number of these transitions need to be taken into account when the limit  $B \rightarrow 0$  is considered, so that the zero field limit is easily treatable [8] on the base of equation (7). Another important feature of equation (7) is that we kept the frequency dependent impurity scattering rate  $\Gamma(\omega)$ , which allows us to investigate its influence on the shape of the Drude peak [16].

Although in our work we will mostly use equation (7) for the numerical computations of the diagonal conductivity, there is a possibility to derive simple approximate expressions for the diagonal and Hall conductivities which turn out to be very useful when one is interested in the resonance peaks of these conductivities in the infrared region.

## 2.2. Landau level index dependent scattering rate and magneto-optical Lorentzian model

For analysing experimental data it is useful to have a magneto-optical Lorentzian model for the complex conductivity,  $\sigma_{\pm}(\Omega) = \sigma_{xx}(\Omega) \pm i\sigma_{xy}(\Omega)$  due to inter- and intraband Landau level transitions [26]. For the case of Dirac fermions it is considered in the appendix, whereof we obtain the complex diagonal conductivity

$$\begin{aligned} \sigma_{xx}(\Omega) = & -\frac{e^2 v_F^2 |eB|}{2\pi c i} \\ & \times \sum_{n=0}^{\infty} \left\{ \left( 1 - \frac{\Delta^2}{M_n M_{n+1}} \right) ([n_F(M_n) - n_F(M_{n+1})] + [n_F(-M_{n+1}) - n_F(-M_n)]) \right. \\ & \times \left( \frac{1}{M_n - M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} - \frac{1}{M_n - M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right) \\ & \times \frac{1}{M_{n+1} - M_n} + \left( 1 + \frac{\Delta^2}{M_n M_{n+1}} \right) ([n_F(-M_n) - n_F(M_{n+1})] \\ & + [n_F(-M_{n+1}) - n_F(M_n)]) \frac{1}{M_{n+1} + M_n} \\ & \left. \times \left( \frac{1}{M_n + M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} - \frac{1}{M_n + M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right) \right\}, \end{aligned} \quad (9)$$

and the complex Hall conductivity

$$\begin{aligned} \sigma_{xy}(\Omega) = & \frac{e^2 v_F^2 eB}{2\pi c} \sum_{n=0}^{\infty} ([n_F(M_n) - n_F(M_{n+1})] - [n_F(-M_{n+1}) - n_F(-M_n)]) \\ & \times \left\{ \left( 1 - \frac{\Delta^2}{M_n M_{n+1}} \right) \frac{1}{M_{n+1} - M_n} \right. \\ & \times \left( \frac{1}{M_n - M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} + \frac{1}{M_n - M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right) \\ & - \left( 1 + \frac{\Delta^2}{M_n M_{n+1}} \right) \frac{1}{M_{n+1} + M_n} \\ & \left. \times \left( \frac{1}{M_n + M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} + \frac{1}{M_n + M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right) \right\}. \end{aligned} \quad (10)$$

In deriving equations (9), (10) we assumed that the impurity scattering rate is independent of the energy  $\omega$ , but kept its dependence on the Landau level index  $n$ . This assumption allows us to eliminate the integration over  $\omega$  which is present in equation (7). To preserve the Landau index dependence, the sum over transitions between neighbouring Landau levels is retained in equations (9) and (10).

Based on equations (9) and (10) one can easily write down separate expressions for  $\text{Re } \sigma_{xx}(\Omega)$ ,  $\text{Im } \sigma_{xx}(\Omega)$ ,  $\text{Re } \sigma_{xy}(\Omega)$  and  $\text{Im } \sigma_{xy}(\Omega)$  and verify that diagonal and off-diagonal conductivities satisfy Kramers–Kronig relations. A big advantage of equations (9) and (10) is that they are more suitable for numerical computations and it is sufficient to include only a few terms in the sum even in relatively low magnetic field. Also they are useful for the description of the resonance peaks when the Landau level index dependence is more important than the energy dependence of the scattering rate which is included in equation (7).

All figures in this paper were computed for  $\Gamma = \text{const}$ , so for calculation of the diagonal conductivity we used equation (7) and checked that it gives almost identical results to equation (9). Since the corresponding full expression for the optical Hall conductivity derived in [8] is rather difficult for numerical calculations, in the present paper we compute all results for the Hall conductivity using equation (10). In the case of Landau level index independent width,  $\Gamma_n = \text{const}$ , equations (9) and (10) acquire an even simpler form,

$$\begin{aligned} \sigma_{xx}(\Omega) = & \frac{e^2 v_F^2 |eB| (\Omega + 2i\Gamma)}{\pi c i} \\ & \times \sum_{n=0}^{\infty} \left\{ \left( 1 - \frac{\Delta^2}{M_n M_{n+1}} \right) \frac{[n_F(M_n) - n_F(M_{n+1})] + [n_F(-M_{n+1}) - n_F(-M_n)]}{(M_{n+1} - M_n)^2 - (\Omega + 2i\Gamma)^2} \right. \\ & \times \frac{1}{M_{n+1} - M_n} + \left( 1 + \frac{\Delta^2}{M_n M_{n+1}} \right) \\ & \left. \times \frac{[n_F(-M_n) - n_F(M_{n+1})] + [n_F(-M_{n+1}) - n_F(M_n)]}{(M_{n+1} + M_n)^2 - (\Omega + 2i\Gamma)^2} \frac{1}{M_{n+1} + M_n} \right\} \end{aligned} \quad (11)$$

and

$$\begin{aligned} \sigma_{xy}(\Omega) = & -\frac{e^2 v_F^2 eB}{\pi c} \sum_{n=0}^{\infty} ([n_F(M_n) - n_F(M_{n+1})] - [n_F(-M_{n+1}) - n_F(-M_n)]) \\ & \times \left\{ \left( 1 - \frac{\Delta^2}{M_n M_{n+1}} \right) \frac{1}{(M_{n+1} - M_n)^2 - (\Omega + 2i\Gamma)^2} \right. \\ & \left. + \left( 1 + \frac{\Delta^2}{M_n M_{n+1}} \right) \frac{1}{(M_{n+1} + M_n)^2 - (\Omega + 2i\Gamma)^2} \right\}. \end{aligned} \quad (12)$$

One can see that the conductivity  $\sigma_{xx}(\Omega, \mu)$  is an even function of  $\mu$  while  $\sigma_{xy}(\Omega, \mu)$  is an odd one.

### 2.3. Low field limit of the magneto-optical Lorentzian model

Now we check that equations (11), (12) reproduce correctly the limit  $B \rightarrow 0$ . Introducing the continuum variable  $\omega$  instead of  $M_n$  given by equation (5) and replacing the sum over  $n$  by the integral, we obtain

$$\begin{aligned} \sigma_{xx}(\Omega) = & -\frac{2ie^2(\Omega + 2i\Gamma)}{h} \left[ \frac{1}{(\Omega + 2i\Gamma)^2} \int_{\Delta}^{\infty} d\omega \frac{\omega^2 - \Delta^2}{\omega} \left( \frac{\partial n_F(\omega)}{\partial \omega} - \frac{\partial n_F(-\omega)}{\partial \omega} \right) \right. \\ & \left. - \int_{\Delta}^{\infty} d\omega \frac{\omega^2 + \Delta^2}{\omega^2} \frac{n_F(-\omega) - n_F(\omega)}{(\Omega + 2i\Gamma)^2 - 4\omega^2} \right] \end{aligned} \quad (13)$$

and

$$\begin{aligned} \sigma_{xy}(\Omega) = & \frac{e^2 v_F^2 eB}{\pi c} \int_{\Delta}^{\infty} d\omega \left( \frac{\partial n_F(\omega)}{\partial \omega} + \frac{\partial n_F(-\omega)}{\partial \omega} \right) \\ & \times \left[ -\frac{\omega^2 - \Delta^2}{\omega^2} \frac{1}{(\Omega + 2i\Gamma)^2} + \frac{\omega^2 + \Delta^2}{\omega^2} \frac{1}{4\omega^2 - (\Omega + 2i\Gamma)^2} \right], \end{aligned} \quad (14)$$

where we restored Planck constant  $h = 2\pi\hbar$  in the overall prefactor. Here the terms with the factor  $(\omega^2 - \Delta^2)/\omega^2$  are intraband and the terms containing the factor  $(\omega^2 + \Delta^2)/\omega^2$  are

interband. The expressions (13) and (14) are obtained under the condition  $\sqrt{\hbar|eB|v_F^2/c} \ll \Gamma$ . The intraband term of equation (13) can be written in the familiar Drude form

$$\sigma_{xx}^{\text{Drude}}(\Omega) = \frac{2e^2}{h} \int_{-\infty}^{\infty} d\omega \left( -\frac{\partial n_F(\omega)}{\partial \omega} \right) \frac{1}{2\Gamma - i\Omega} \frac{(\omega^2 - \Delta^2)\theta(\omega^2 - \Delta^2)}{|\omega|} \quad (15)$$

which for  $\Delta = 0$  reduces to equation (2) of [16]. The whole expression (13), which includes the interband term, also reduces to the other limiting cases considered in [16, 27]. In particular, an unusual feature of graphene is that in the high frequency limit the interband contribution is a constant,

$$\sigma_{xx}(\Omega) \simeq \frac{\pi e^2}{2h}, \quad \Omega \gg \mu, \Delta, T. \quad (16)$$

Here to rely on the linearized Dirac approximation we assumed that  $\Omega$  is still well below a large band edge.

The real part of the Hall conductivity (14) for  $\Omega = \Delta = T = 0$  reduces to the expression

$$\sigma_{xy}(\Omega = 0) = -\frac{e^2 v_F^2 e B \operatorname{sgn} \mu}{4\pi c \Gamma^2}, \quad \sqrt{\hbar|eB|v_F^2/c} \ll \Gamma \ll |\mu| \quad (17)$$

which is in agreement with equation (4.3) of [8]. On the other hand, in the high frequency limit equation (14) gives

$$\sigma_{xy}(\Omega) = \frac{e^2 v_F^2 e B}{\pi c (\hbar \Omega)^2} \left[ \tanh \frac{\mu + \Delta}{2T} + \tanh \frac{\mu - \Delta}{2T} \right], \quad \Omega \rightarrow \infty. \quad (18)$$

This behaviour also follows from equation (10), which is valid in an arbitrary field  $B$ . Interestingly, expression (18) is sensitive to the relationship between  $|\mu|$  and  $\Delta$  and this feature can be used for the gap detection (see the discussion of figures 4 and 9 below). Using equations (16) and (18) we obtain the weak field optical Hall resistivity

$$\begin{aligned} \rho_{xy}(\Omega) &= -\frac{\sigma_{xy}(\Omega)}{\sigma_{xx}^2(\Omega) + \sigma_{xy}^2(\Omega)} \\ &= -\frac{16v_F^2 B}{\pi e c \Omega^2} \left[ \tanh \frac{\mu + \Delta}{2T} + \tanh \frac{\mu - \Delta}{2T} \right], \quad \Omega \gg \mu, \Delta, T. \end{aligned} \quad (19)$$

Accordingly, the high frequency optical Hall coefficient for  $\Delta = 0$  and  $T \rightarrow 0$

$$R_H(\Omega) = \frac{\rho_{xy}(\Omega)}{B} = -\frac{32v_F^2}{\pi e c \Omega^2} \operatorname{sgn}(\mu) \quad (20)$$

contains the information on the value of the Fermi velocity in graphene.

Finally, the imaginary part of  $\sigma_{xy}(\Omega)$  which follows from equation (14) is given by the expression

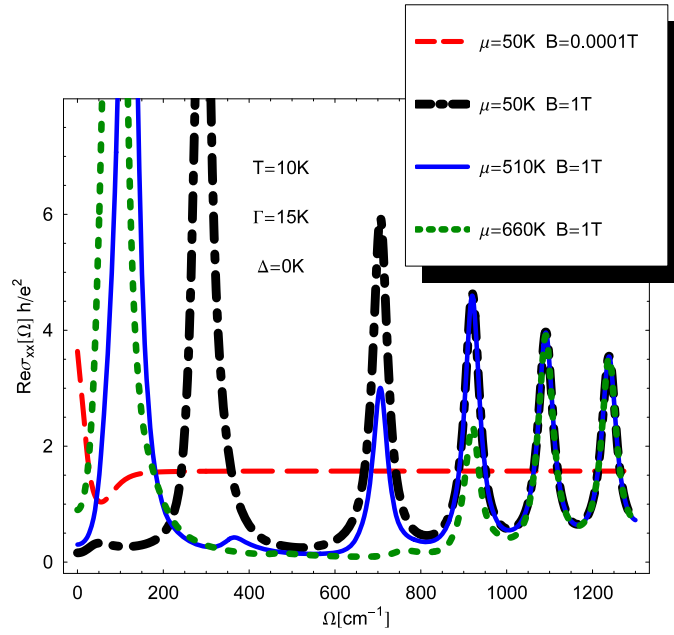
$$\begin{aligned} \operatorname{Im} \sigma_{xy}(\Omega) &= \frac{4e^2 v_F^2 e B \Gamma \Omega}{\pi c} \int_{\Delta}^{\infty} d\omega \left( \frac{\partial n_F(\omega)}{\partial \omega} + \frac{\partial n_F(-\omega)}{\partial \omega} \right) \\ &\quad \times \left[ \frac{1}{(\Omega^2 - 4\Gamma^2)^2 + 16\Omega^2 \Gamma^2} + \frac{1}{(4\omega^2 - \Omega^2 + 4\Gamma^2)^2 + 16\Omega^2 \Gamma^2} \right] \end{aligned} \quad (21)$$

which in the high frequency limit becomes

$$\operatorname{Im} \sigma_{xy}(\Omega) \simeq -\frac{4e^2 v_F^2 e B \Gamma}{\pi c (\hbar \Omega)^3} \left( \tanh \frac{\mu + \Delta}{2T} + \tanh \frac{\mu - \Delta}{2T} \right), \quad \Omega \rightarrow \infty. \quad (22)$$

The last equation shows that in this limit the real part of  $\sigma_{xy}$  given by equation (18) is the leading term.





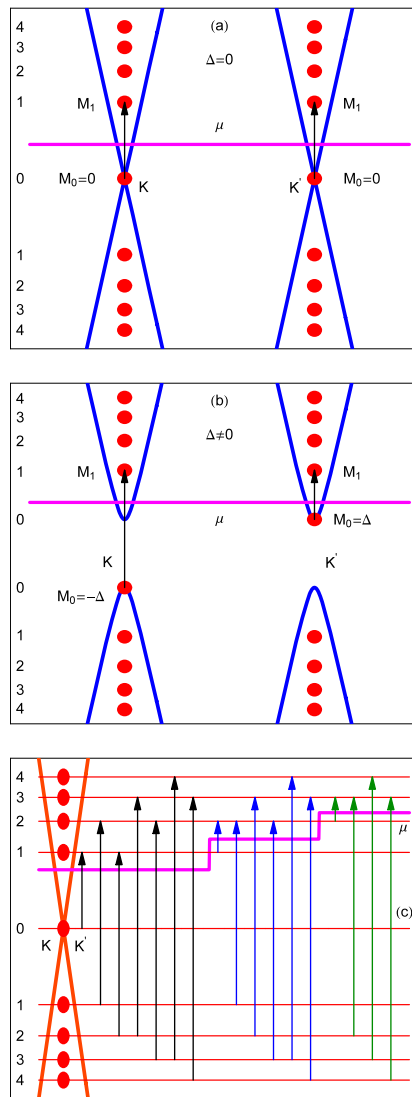
**Figure 1.** The real part of the longitudinal conductivity,  $\text{Re } \sigma_{xx}(\Omega)$ , in units of  $e^2/h$  versus frequency  $\Omega$  in  $\text{cm}^{-1}$  for temperature  $T = 10$  K and scattering rate  $\Gamma = 15$  K. Long dashed, the chemical potential  $\mu = 50$  K and the magnetic field  $B = 10^{-4}$  T; dash-dotted,  $\mu = 50$  K and  $B = 1$  T; solid,  $\mu = 510$  K and  $B = 1$  T; short dashed,  $\mu = 660$  K and  $B = 1$  T.

### 3. Results for optical conductivity

#### 3.1. Diagonal conductivity

In figure 1 we show the results based on a numerical evaluation of the full equation (7) for the real part of the longitudinal conductivity  $\text{Re } \sigma_{xx}(\Omega)$  in units  $e^2/h$  as a function of frequency in units of  $\text{cm}^{-1}$ . Except for the long dashed (red) curve which was obtained in the limit of vanishing external magnetic field (namely  $B = 10^{-4}$  T) and is included for comparison, the other three curves are for  $B = 1$  T. They differ in value of the chemical potential  $\mu$ . In all cases the temperature  $T = 10$  K, the impurity scattering  $\Gamma = 15$  K and the excitonic gap  $\Delta = 0$ . For reference in scrutinizing the curves, the frequency of the  $n = 1$  Landau level,  $\Omega_1 = M_1(\Delta = 0) = 294 \text{ cm}^{-1}$  (423 K),  $\Omega_2 = M_2(\Delta = 0) = 415.8 \text{ cm}^{-1}$  (598 K) and  $\Omega_3 = M_3(\Delta = 0) = 509.2 \text{ cm}^{-1}$  (733 K), so that for the dash-dotted (black) curve  $\mu$  falls below the energy of the  $n = 1$  level (see the left side of figure 2(c))<sup>4</sup>, for the solid (blue) curve it falls between  $n = 1$  and 2 levels (see the middle of figure 2(c)) and for the short dashed (green) curve it is between  $n = 2$  and 3 (right side of figure 2(c)). The energies of the peaks are  $M_1$ ,  $M_1 + M_2$ ,  $M_2 + M_3$  etc and  $M_2 - M_1$ ,  $M_3 - M_2$  etc. Note that the position and the intensity of the last two peaks in figure 1 (largest frequency  $\Omega$ ) remain the same for all chosen values of the chemical potential  $\mu$ . When  $\mu$  falls between  $M_2$  and  $M_3$  (short dashed (green) curve) the intensity in the third last peak has dropped to half the value it has in the solid (blue) curve, while the fourth last peak has merged into a low intensity background, as has the fifth last, which is seen only in the dash-dotted curve. Also a new peak has appeared at  $M_3 - M_2$  which was

<sup>4</sup> We recall that the conversion rule from the frequency in  $\text{cm}^{-1}$  to the energy in kelvin is  $\Omega (\text{K}) = 1.4387 \text{ K cm } \Omega (\text{cm}^{-1})$ .



**Figure 2.** (a) Schematic representation of the two pairs of Dirac cones with apex at points  $\mathbf{K}$  (left) and  $\mathbf{K}'$  (right) in the graphene Brillouin zone. The energies of the Landau levels are shown for index  $n = 0, \dots, 4$  as solid (red) circles for both positive and negative Dirac cones. The transition from  $n = 0$  to 1 across the chemical potential shown as a thick horizontal (violet) line is for the case  $\Delta = 0$ . (b) The same as (a) but now there is a finite excitonic gap  $\Delta$ . (c) The pair of cones at points  $\mathbf{K}$  and  $\mathbf{K}'$  in the Brillouin zone (see panels (a) and (b)) is combined. Vertical arrows show allowed optical transitions between Landau levels for three values of the chemical potential. On the left are the transitions when  $\mu$  is between  $n = 0$  and 1, in the middle between  $n = 1$  and 2 and on the right between  $n = 2$  and 3. The first line on the left is different in that it is both inter (between two separate cones) and intra (within a given cone). The first line in all other series is intra and the shortest inter appears only once, while all others appear twice. Note that from equation (11)  $\sigma_{xx}(\Omega, \mu)$  is even in  $\mu$ , while from equation (12)  $\sigma_{xy}(\Omega, \mu)$  is odd in  $\mu$ .

not present in the dash-dotted curve. Similarly for  $\mu$  between  $M_1$  and  $M_2$  (solid (blue) curve) the intensity of the fourth highest energy peak has dropped to half the intensity it has in the

dash-dotted curve and the peak at  $M_1$  is entirely missing, having merged into the low intensity background. A new peak has appeared at  $M_2 - M_1$ . Finally, when  $\mu$  is below  $M_1$  (dash-dotted curve) there is no peak below  $M_1$  and the line at  $M_1$  always has full intensity. *Whatever the value of  $\mu$ , this line will never be seen to half its intensity and this is the hallmark of the Dirac nature of the quasiparticles.* A schematic diagram which helps us understand the behaviour of the absorption lines that we have just described is given in figure 2(c). On the left of the figure we show the energies of the Landau levels  $E_n = \pm M_n$  of equation (5) as solid (red) dots along with their quantum numbers  $n = 0, 1, 2, \dots$ . The Dirac cones, which come in pairs, with positive and negative energies, and exist at  $\mathbf{K}$  and  $\mathbf{K}'$  points in the graphene Brillouin zone, are also shown. The three values of chemical potential considered in figure 2(c) are shown in heavy solid (violet) horizontal lines. The possible optical transitions in each case are indicated as vertical arrows and connect levels  $n$  to  $n \pm 1$  only. Moving from left to right we see first a single transition from  $E_0$  to  $E_1 = M_1$ , then a pair of interband transitions from  $E_1 = -M_1$  to  $E_2 = M_2$  and  $E_2 = -M_2$  to  $E_1 = M_1$  followed by another pair from  $E_2 = -M_2$  to  $E_3 = M_3$  and  $E_3 = -M_3$  to  $E_2 = M_2$  etc. For the middle set of lines the first is an intraband transition from  $E_1 = M_1$  to  $E_2 = M_2$  followed by a single interband transition from  $E_1 = -M_1$  to  $E_2 = M_2$  and then a pair from  $E_2 = -M_2$  and  $E_3 = M_3$  and  $E_3 = -M_3$  to  $E_2 = M_2$  etc. Finally, in the set of transitions on the right of the figure there is an intraband transition from  $E_2 = M_2$  to  $E_3 = M_3$  followed by a single interband transition from  $E_2 = -M_2$  to  $E_3 = M_3$  and a pair from  $E_3 = -M_3$  to  $E_4 = M_4$  and  $E_4 = -M_4$  to  $E_3 = M_3$  etc. This is precisely the pattern we have seen in figure 1. We note one more aspect of the anomalous line on the far left of the figure for the transition from the  $n = 0$  to  $n = 1$  Landau level. It is the only line which cannot be unambiguously assigned to inter or intraband transition because the state at  $n = 0$  falls at the apex of the Dirac cones, where positive and negative energy cones meet, and hence they share this state equally. This is further illustrated in figures 2(a) and (b), where two sets of cones at  $\mathbf{K}$  and  $\mathbf{K}'$  are shown separately for the case of  $\Delta = 0$  in frame (a) and finite excitonic gap in frame (b). In this second case we see clearly that points  $\mathbf{K}$  and  $\mathbf{K}'$  react differently under a finite magnetic field. For the cone on the left the  $n = 0$  level has moved to energy  $E_0 = -\Delta$  and for the cone on the right it has moved to  $E_0 = \Delta$ . Note that for the value of chemical potential shown as a solid horizontal (violet) line the  $E_0 = -\Delta$  to  $E_1$  transition (vertical arrow) on the left is now definitely interband and the  $E_0 = \Delta$  to  $E_1$  on the right is intraband. The ambiguity in designation of the  $n = 0$  level present in the top frame (a) of figure 2 is lifted when the gap becomes finite.

The rather complicated pattern of behaviour just described can be understood simply from the mathematics of the previous section in the limit  $\Gamma \rightarrow 0$  and  $T \rightarrow 0$ . Taking  $\Delta = 0$ ,  $\mu \geq 0$  and  $\Omega \geq 0$ , one obtains from equation (A.12)

$$\begin{aligned} \text{Re } \sigma_{xx}(\Omega) = \frac{e^2}{h} M_1^2 \frac{\pi}{2} \sum_{n=0}^{\infty} \left\{ [2 - n_F(M_n) - n_F(M_{n+1})] \frac{\delta(M_n + M_{n+1} - \Omega)}{M_n + M_{n+1}} \right. \\ \left. + [n_F(M_n) - n_F(M_{n+1})] \frac{\delta(M_n - M_{n+1} + \Omega)}{M_{n+1} - M_n} \right\}. \end{aligned} \quad (23)$$

For  $\mu \in ]M_N, M_{N+1}[$  the  $T = 0$  thermal factor

$$2 - n_F(M_n) - n_F(M_{n+1}) = \begin{cases} 0 & \text{for } n < N, \\ 1 & \text{for } n = N, \\ 2 & \text{for } n > N, \end{cases} \quad (24)$$

while

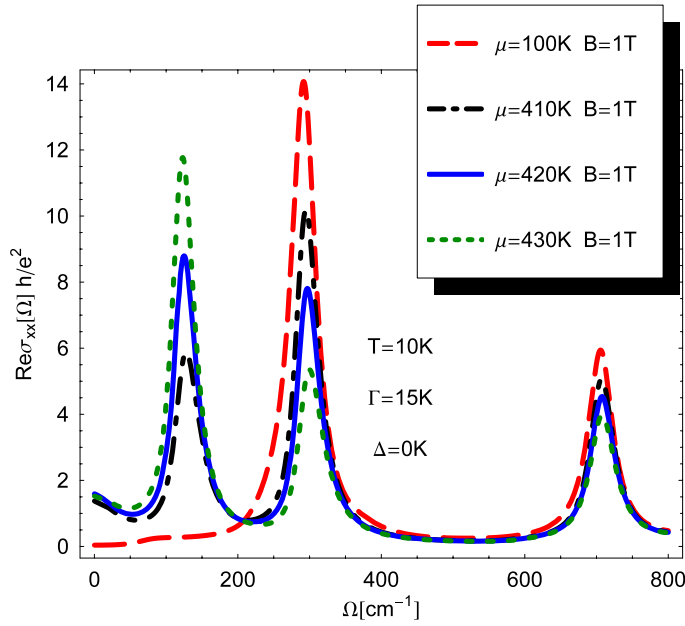
$$n_F(M_n) - n_F(M_{n+1}) = \begin{cases} 1 & \text{for } n = N, \\ 0 & \text{for } n \neq N. \end{cases} \quad (25)$$

The line in  $\text{Re } \sigma_{xx}(\Omega)$  with the frequency  $M_n + M_{n+1}$  occurs only for  $n \geq N$  with the others missing. Further, the line with  $n = N$  has half of the weight it would have if it occurred in another case, namely,  $1/(M_n + M_{n+1})$ , while lines for  $n > N$  have full weight  $2/(M_n + M_{n+1})$ . Note that the lines with the frequency equal to the difference in Landau level energies have weight  $1/(M_{n+1} - M_n)$ . These lines are always present except for the case when  $\mu$  falls below  $M_1$ , when the single line at  $M_1$  has weight  $2/M_1$ . In summary, we have seen in the above discussion that as  $\mu$  moves through higher and higher values of  $M_N$  the lines below  $N$  disappear into the background, with a new line appearing at  $M_{N+1} - M_N$ . Further, the line at  $n = N$  loses half its intensity while the others remain unaltered, except for the special case when chemical potential falls below the Landau level  $M_1$ .

The pattern would be quite different if instead of  $M_n \sim \sqrt{n}$  the Landau level quantization were Schrödinger-like  $M_n = \omega_c(n + 1/2)$ , with  $\omega_c$  being the cyclotron frequency. In this case the position of the line corresponding to the difference in Landau level energies (intra-band) never shifts in energy. Further, the lines corresponding to the sum of the Landau level energies (inter-band) fall at regular energy intervals  $2\omega_c(n + 1)$ , namely  $2\omega_c, 4\omega_c, 6\omega_c, \dots$ . Furthermore, as  $\mu$  increases through the energies of the various Landau levels all lines halve their intensity before fading into the background and for  $\mu$  below the first level there is no line at  $\omega_c$ . There is also no ambiguity about whether a line is intra or inter-band as no level is shared between upper and lower cone.

In field effect devices the chemical potential in a graphene sheet can be changed by applying a gate voltage and this may be an ideal way to observe the effects just described. However, there is an alternative way to see the same effects. For any fixed value of chemical potential, the external magnetic field can be selected in such a way that  $\mu$  falls below the first Landau level or in between  $n = 1$  and 2 etc. For appropriate choices of  $B$  [28] the curves for  $\text{Re } \sigma_{xx}(\Omega)$  can be made to behave exactly as in figure 1. To see this, it is important that  $\Omega$  be divided by  $\sqrt{\hbar e B v_F^2 / c}$ , so that the lines remain fixed in normalized frequency and the vertical scale be divided by the same factor and multiplied by  $\Gamma$  to keep the dimensions the same. When this is done, the same pattern as seen in figure 1 emerges for this configuration corresponding to fixed  $\mu$  with several well chosen values of  $B$ .

We have found that the curves of figure 1 change very little as the chemical potential is varied within the limited range  $M_N$  to  $M_{N+1}$ . We have also verified that for the parameters used here,  $T = 10$  K,  $\Gamma = 15$  K, the crossover from half intensity in the line  $M_N + M_{N+1}$  to near zero, i.e. merging into the background, occurs rather abruptly in an energy range set by  $T$  and/or  $\Gamma$ . This is illustrated in figure 3, where we consider how the complete disappearance of the peak at  $\Omega = \Omega_1 = 294 \text{ cm}^{-1}$  (420 K) and the depletion of the next higher peak at  $710 \text{ cm}^{-1}$  towards half its initial value proceeds as the chemical potential  $\mu$  crosses through the energy of the  $n = 1$  Landau level ( $\Omega_1 = M_1(\Delta = 0)$ ). The long dashed curve (red) is for  $\mu = 100$  K chosen to be well away from the crossover point of 420 K and is shown for comparison. The dash-dotted curve (black) is for  $\mu = 410$  K slightly below the crossover point, the solid curve (blue) is for  $\mu = 420$  K just at the crossover energy and the short dashed curve (green) is for  $\mu = 430$  K, which spans  $\pm 10$  K on either side of  $\Omega_1$ , which is much less than the level width of  $2\Gamma = 30$  K. As  $\mu$  increases through 410 K we note the growth of the peak at  $\Omega = 122 \text{ cm}^{-1}$ , the depletion towards zero value of the peak at  $294 \text{ cm}^{-1}$  and that towards half its initial value of the next higher peak at  $710 \text{ cm}^{-1}$ . The complete transfer of

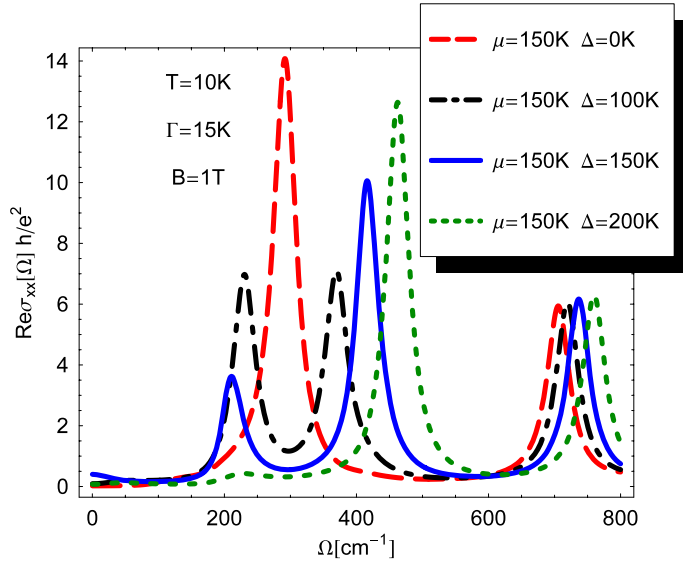


**Figure 3.** Real part of the longitudinal conductivity,  $\text{Re } \sigma_{xx}(\Omega)$  in units of  $e^2/h$  versus frequency  $\Omega$  in  $\text{cm}^{-1}$  for temperature  $T = 10$  K, scattering rate  $\Gamma = 15$  K and magnetic field  $B = 1$  T for four values of chemical potential. Long dashed,  $\mu = 100$  K; dash-dotted,  $\mu = 410$  K; solid,  $\mu = 420$  K; short dashed,  $\mu = 430$  K.

spectral weight between the various peaks is completed for a rather small region of chemical potential variation about  $\Omega_1 = 294 \text{ cm}^{-1}$  (420 K) of order 30 K (not shown in the figure). After the crossover is complete, the pattern of the spectral weight distribution will remain unchanged until the chemical potential becomes close to the energy of the next Landau level. We would like to stress that although figure 3 is plotted using the full expression (7) we have verified that the results obtained using equations (A.12) and (9) are practically identical.

Returning to equations (23)–(25) we note that the optical spectral weight under a given inter- or intraband line varies as the square root of  $B$  (see equation (32) and the discussion associated with it). This has been verified in the recent experiment of Sadowski *et al* [29] on ultrathin epitaxial graphite samples [30, 31].

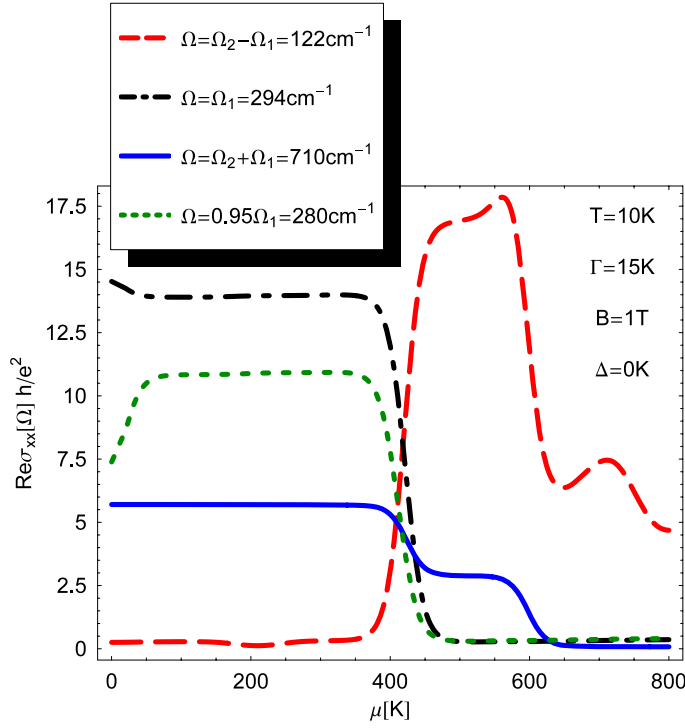
The effect of an excitonic gap  $\Delta$  on the optical spectral weight distribution in a 1 T magnetic field is illustrated in figure 4. Here as in the previous figures  $\Gamma = 15$  K and  $T = 10$  K. The chemical potential is set at 150 K in all cases. The long dashed (red), dash-dotted (black), solid (blue) and short dashed (green) curves are for  $\Delta = 0, 100, 150$  and 200 K respectively. The curve for  $\Delta = 0$  is for reference. We note that for finite  $\Delta = 100$  K the line at  $\Omega_1 = M_1(\Delta = 0) = 294 \text{ cm}^{-1}$  splits into two peaks (dash-dotted (black) curve). The lower peak is at energy  $M_1(\Delta) - M_0(\Delta) = \sqrt{\Omega_1^2 + \Delta^2} - \Delta$ , while the upper peak is at  $M_1(\Delta) + M_0(\Delta) = \sqrt{\Omega_1^2 + \Delta^2} + \Delta$ . These two transitions for this value of chemical potential are illustrated in figure 2, middle frame (b) (see the arrows). Additional transitions not shown in this frame can, of course, occur, but these will have higher energy. For  $\Delta = 150 \text{ K} = \mu$  the  $n = 0$  state shown on the right-hand cone of figure 2(b) is occupied with a probability 1/2 at  $T = 0$ ,  $\Gamma = 0$ , and a transition from the  $n = 1$  lower cone to the  $n = 0$  upper cone



**Figure 4.** Real part of the longitudinal conductivity,  $\text{Re } \sigma_{xx}(\Omega)$ , in units of  $e^2/h$  versus frequency  $\Omega$  in  $\text{cm}^{-1}$  for temperature  $T = 10$  K,  $\Gamma = 15$  K,  $B = 1$  T and chemical potential  $\mu = 150$  K for four values of the excitonic gap  $\Delta$ . Long dashed,  $\Delta = 0$  K; dash-dotted,  $\Delta = 100$  K; solid,  $\Delta = 150$  K; short dashed,  $\Delta = 200$  K.

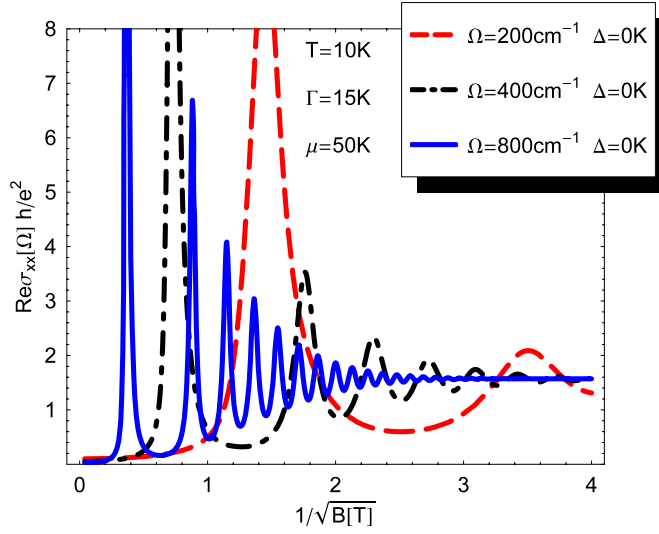
is possible, as it is from the  $n = 0$  upper cone to  $n = 1$  of the same cone and each must be weighted by a factor  $1/2$ . The first has energy  $\sqrt{\Omega_1^2 + \Delta^2} + \Delta$ , while the second has energy  $\sqrt{\Omega_1^2 + \Delta^2} - \Delta$ . There is an additional transition coming from the second cone on the left-hand side of the figure of energy  $\sqrt{\Omega_1^2 + \Delta^2} + \Delta$ . Thus the spectral intensity of the lower energy line in the solid curve (blue) is lower than that of the higher energy line by a factor of 3. Finally, for  $\Delta = 200$  K ( $\Delta > \mu$ ) the energies of the lowest transition possible in both left and right side cones are the same, equal to  $\sqrt{\Omega_1^2 + \Delta^2} + \Delta$ , and so there is only one line in the dashed (green) curve. This pattern of behaviour should allow one to measure the occurrence of an excitonic gap catalysed by the magnetic field. We note that the split peak in figure 4 with finite  $\Delta$  acquires a slight asymmetry because of the factor  $1 \pm \Delta^2/M_n M_{n+1}$  of equation (A.12), which is different for inter- and intraband transitions.

So far we have shown the results for fixed values of chemical potential and magnetic field as a function of photon energy. It is also of interest to fix the photon frequency and sweep either chemical potential ( $\mu$ ) or field ( $B$ ) [32]. We begin with the first of these two possibilities and this is illustrated in figure 5. Four well chosen frequencies are selected, namely  $\Omega = \Omega_2 - \Omega_1 = 122 \text{ cm}^{-1}$ , long dashed curve (red);  $\Omega = \Omega_1 = 294 \text{ cm}^{-1}$ , dash-dotted curve (black);  $\Omega = \Omega_2 + \Omega_1 = 710 \text{ cm}^{-1}$ , solid curve (blue), and  $\Omega = 0.95\Omega_1 = 280 \text{ cm}^{-1}$ , short dashed curve (green). Taking these in order, we note that the long dashed curve (red) is near zero until  $\mu$  reaches the value of 420 K, at which point it increases rapidly, reaching a plateau at  $\text{Re } \sigma_{xx}(\Omega) \approx 17.5e^2/h$  (the height being set by the height of the peak in the solid (blue) curve in figure 1), after which it drops rapidly as  $\mu$  goes through 595 K. It does not drop down all the way to zero, however, as it continues to sample parts of the peaks at the difference in frequencies  $M_{n+1} - M_n$  until these move below the sampling optical frequency set at  $122 \text{ cm}^{-1}$ . The next



**Figure 5.** Real part of the longitudinal conductivity,  $\text{Re } \sigma_{xx}(\Omega)$  in units of  $e^2/h$  versus the chemical potential  $\mu$  in K for  $T = 10$  K,  $\Gamma = 15$  K,  $B = 1$  T and  $\Delta = 0$ . Four frequencies  $\Omega$  are considered, long dashed  $\Omega = 122$   $\text{cm}^{-1}$ , dash-dotted  $\Omega = 294$   $\text{cm}^{-1}$ , solid  $\Omega = 710$   $\text{cm}^{-1}$ , short dashed  $\Omega = 0.95\Omega_1 = 280$   $\text{cm}^{-1}$ . The first three frequencies correspond to  $\Omega = \Omega_2 - \Omega_1$ ,  $\Omega_1$  and  $\Omega_2 + \Omega_1$ , respectively, for the parameters used.

optical frequency chosen at  $\Omega = \Omega_1$ , short-long dashed (black) curve, begins by sampling the equivalent curve in figure 1, where the peak has height  $\sim 14e^2/h$ . However, when  $\mu$  crosses the energy of the first Landau level at 420 K this peak disappears and the curve drops to zero. No other level crosses this frequency again. The short dashed curve is for  $\Omega = 0.95\Omega_1$ . It shows very much the same behaviour as the previous case, but its plateau height is a little smaller because the peak (dash-dotted curve of figure 1) is sampled not at its centre, but rather slightly below its maximum value. The final solid (blue) curve is for  $\Omega = \Omega_2 + \Omega_1$ . In this case a first plateau is seen at small  $\mu$  with height  $\sim 5.5e^2/h$ , which is the height of the second peak in the dash-dotted (black) curve of figure 1. It drops to half of this value as  $\Omega$  crosses 420 K and then near zero as  $\Omega$  crosses 595 K. All features of these curves can be traced to the corresponding behaviour of the curves of figure 1. An additional feature of these curves is now described. Note that as the chemical potential gets small the black (dash-dotted) curve increases slightly, while the green (dotted) curve drops. As we have described in connection with figure 3 the line shapes do depend on the value of  $\mu$  if it falls within order  $\Gamma$  and/or  $T$  of a Landau level energy  $M_n$ . For  $\mu$  near zero, the first peak height in the black (dash-dotted) curve of figure 1 increases by  $\sim 5\%$  and its width is also slightly narrowed. This leads to a slight increase in peak height at  $\Omega = \Omega_1 = 294$   $\text{cm}^{-1}$  monitored in the black (dash-dotted) curve of figure 5 and a reduction in the green (dotted) curve, which monitors the height of the curve slightly off the peak at  $\Omega = 0.95\Omega_1 = 280$   $\text{cm}^{-1}$ .



**Figure 6.** Real part of the longitudinal conductivity,  $\text{Re } \sigma_{xx}(\Omega)$  at three frequencies (long dashed,  $200 \text{ cm}^{-1}$ ; dash-dotted,  $400 \text{ cm}^{-1}$ ; solid,  $800 \text{ cm}^{-1}$ ) as a function of the inverse of the square root of the magnetic field,  $1/\sqrt{B}$ , with  $B$  in T. The other parameters are  $T = 10 \text{ K}$ ,  $\Gamma = 15 \text{ K}$  and  $\mu = 50 \text{ K}$ .

Another set of useful curves when considering possible experimental configurations is to fix the frequency of the light as well as the chemical potential and sweep the magnetic field. Results are shown in figure 6 for three cases,  $\Omega = 200 \text{ cm}^{-1}$ , long dashed (red) curve, the dash-dotted (black) curve for  $\Omega = 400 \text{ cm}^{-1}$  and solid (blue) for  $\Omega = 800 \text{ cm}^{-1}$ . In all cases  $T = 10 \text{ K}$ ,  $\Gamma = 15 \text{ K}$ ,  $\mu = 50 \text{ K}$  and the excitonic gap  $\Delta$  is taken to be zero. On the horizontal axis we plotted  $1/\sqrt{B} \text{ (T)}$ , the inverse square root of the magnetic field in tesla, so that fields below 1 T fall above one on this scale. The pattern of oscillations is perhaps more complex than in previous curves, but can easily be traced out from a knowledge of such curves for  $\text{Re } \sigma_{xx}(\Omega)$  at different values of  $B$ . As an example consider the case  $\Omega = 200 \text{ cm}^{-1}$ . The value of  $\text{Re } \sigma_{xx}(\Omega)h/e^2$  at  $1/\sqrt{B} \text{ (T)} = 1$  is just that of the dash-dotted (black) curve of figure 1 read at  $\Omega = 200 \text{ cm}^{-1}$ . As  $B$  is decreased from its value 1 T, the first peak in the dash-dotted curve in figure 1 will move to lower frequency and so increases significantly the value of longitudinal conductivity at  $\Omega = 200 \text{ cm}^{-1}$ . Its peak will cross this reference energy for  $1/\sqrt{B} \text{ (T)} \simeq 1.47$  and this produces the first peak in the long dashed (red) curve of figure 6. As  $B$  is decreased further, below 1 T, the second peak in the dash-dotted curve of figure 1 will also move through  $\Omega = 200 \text{ cm}^{-1}$ . This occurs for  $1/\sqrt{B} \text{ (T)} \simeq 3.5$ , where a second lower intensity peak is seen in the long dashed (red) curve of figure 6. The other curves of figure 6 can be traced out from similar considerations based on figure 1 with attention paid to the evolution of these curves with changing value of  $B$ .

### 3.2. Spectral weight

An interesting quantity to consider is the optical spectral weight that falls between  $\Omega = 0$  and  $\Omega = \Omega_m$  with  $\Omega_m$  a variable upper limit in the integral

$$W(\Omega_m) = \int_0^{\Omega_m} d\Omega \text{Re } \sigma_{xx}(\Omega). \quad (26)$$



For the case  $T = \Gamma = \Delta = B = 0$  we have [16]

$$\sigma_{xx}(\Omega) = \frac{\pi e^2 N_f}{h} |\mu| \delta(\Omega) + \frac{\pi e^2 N_f}{4h} \theta \left( \frac{|\Omega|}{2} - |\mu| \right). \quad (27)$$

With  $N_f = 2$  this leads to

$$W(\Omega_m) = \frac{e^2}{h} (\pi |\mu| + \theta(\Omega_m/2 - |\mu|) \pi (\Omega_m/2 - |\mu|)) \simeq \frac{e^2 \pi \Omega_m}{h} \quad \text{for } \Omega_m \gg |\mu|. \quad (28)$$

For finite  $B$ , assuming for simplicity  $M_0 < \mu < M_1$  we get from equation (23)

$$\sigma_{xx}(\Omega) = \frac{e^2 N_f v_F^2 |eB|}{2c} \sum_{n=0}^{\infty} \frac{\delta(M_n + M_{n+1} - \Omega)}{M_n + M_{n+1}}. \quad (29)$$

Accordingly for  $\Delta = 0$

$$\begin{aligned} W(\Omega_m) &= \frac{e^2 N_f v_F^2 |eB|}{2c} \sum_{n=0}^{\infty} \frac{\theta(\Omega_m - M_n - M_{n+1})}{M_n + M_{n+1}} \\ &= \frac{\pi e^2}{h} \sqrt{2\hbar |eB| v_F^2 / c} \sum_{n=0}^N \frac{1}{\sqrt{n} + \sqrt{n+1}}. \end{aligned} \quad (30)$$

For large values  $\Omega_m$  the maximum  $N$  that contributes to (30) can be estimated from  $\Omega_m = M_N + M_{N+1} \simeq 2M_N$  and so  $N = \frac{\Omega_m^2}{8\hbar |eB| v_F^2 / c}$ . But

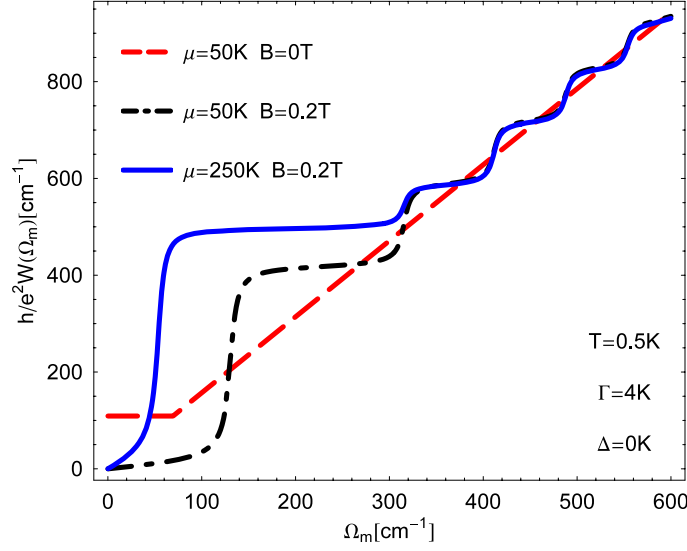
$$\sum_{n=0}^N \frac{1}{\sqrt{n} + \sqrt{n+1}} = \sqrt{N+1} \approx \sqrt{N} \quad \text{for } N \gg 1, \quad (31)$$

and so

$$W(\Omega_m) \simeq \frac{e^2 \pi \Omega_m}{h} \quad (32)$$

which agrees with equation (28) as we would expect. We note that, as we have stated, the area under each line goes as  $\sqrt{B}$ , but the sum over the lines up to  $\Omega_m$  gives another factor  $\sim 1/\sqrt{B}$ , which means that  $W(\Omega_m)$  is independent of  $B$ . In figure 7 we show numerical results for  $(h/e^2)W(\Omega_m)$  in units of  $\text{cm}^{-1}$  as a function of  $\Omega_m$  in  $\text{cm}^{-1}$ . The long dashed (red) curve is computed on the basis of the first equality in equation (28) for  $\mu = 50$  K,  $T = 0$  K, and  $B = 0$  T, and is given for comparison with the two other curves. These curves are obtained by numerical integration of equation (26) with  $\text{Re } \sigma_{xx}(\Omega)$  given by equation (A.12) for the case  $T = 0.5$  K,  $\Gamma = 4$  K,  $\Delta = 0$  and finite  $B = 0.2$  T. The dash-dotted (black) curve is for chemical potential  $\mu = 50$  K, while solid (blue) is for  $\mu = 250$  K. For the first case  $\mu$  is below  $M_1$  at  $131.5 \text{ cm}^{-1}$  and for the second it is above. At small values of  $\Omega_m$  both curves start at zero and remain small until  $\Omega_m$  goes through the first Landau peak in figure 1, at which point it rises sharply and subsequently exhibits a plateau before showing a next sharp rise. For the dash-dotted curve (black) the first rise is at  $131.5 \text{ cm}^{-1}$  (interband line), but for the solid (blue) curve it is at the intraband line ( $54.6 \text{ cm}^{-1}$ ). As  $\Omega_m$  increases to include several peaks in  $\sigma_{xx}(\Omega)$  the curves start to follow more closely the  $B = 0$  T result (long dashed (red) curve). This behaviour can be understood better from a consideration of equation (30). The magnetic field mainly readjusts the available optical spectral weight among the Landau levels below  $\Omega_m$ .

To be specific, we have considered explicitly in this section only the case  $M_0 < \mu < M_1$ . For  $\mu \in ]M_N, M_{N+1}[$  we can show that the missing spectral weight in the lines  $n \leq N$  is all to be found in the single intraband line at  $M_{N+1} - M_N$ . From equation (23) and noting



**Figure 7.** Variation of the optical sum  $W(\Omega_m)$  (multiplied by  $h/e^2$ ) in  $\text{cm}^{-1}$  as a function of  $\Omega_m$  in  $\text{cm}^{-1}$  for three different cases. Long dashed,  $\mu = 50$  K,  $B = 0$  T and  $T = \Gamma = 0$  K (see equation (28)); dash-dotted,  $\mu = 50$  K,  $B = 0.2$  T,  $T = 0.5$  K and  $\Gamma = 4$  K; solid,  $\mu = 250$  K,  $B = 0.2$  T,  $T = 0.5$  K and  $\Gamma = 4$  K.

equations (24) and (25) as  $T \rightarrow 0$ , the optical spectral weight lost in units of  $(e^2/h)M_1(\pi/2)$  is

$$\sum_{n=0}^{N-1} \frac{2}{\sqrt{n+1} + \sqrt{n}} + \frac{1}{\sqrt{N+1} + \sqrt{N}} \equiv \frac{1}{\sqrt{N+1} - \sqrt{N}}. \quad (33)$$

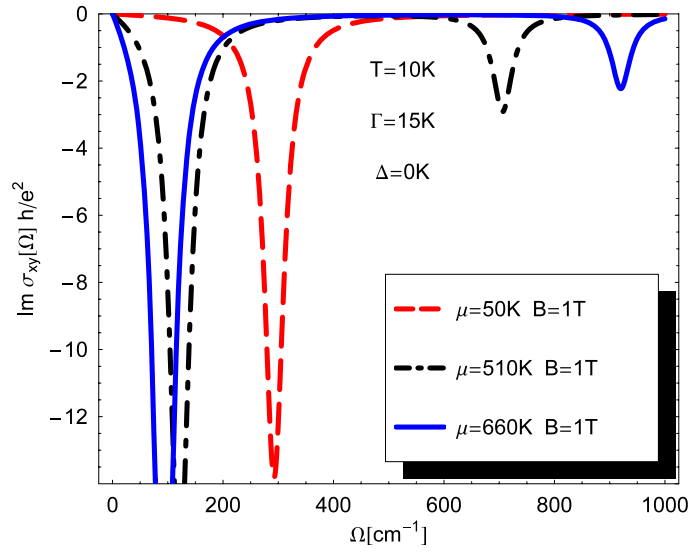
The first term in the left-hand side is the spectral weight from all the lines that have completely disappeared from  $n = 0$  to  $N - 1$ . The second term is from the reduction in intensity by a factor of 1/2 of the line at  $n = N$ . The quantity in the right-hand side is the optical weight of the intraband line which has picked up all of the lost intensity.

### 3.3. Hall conductivity

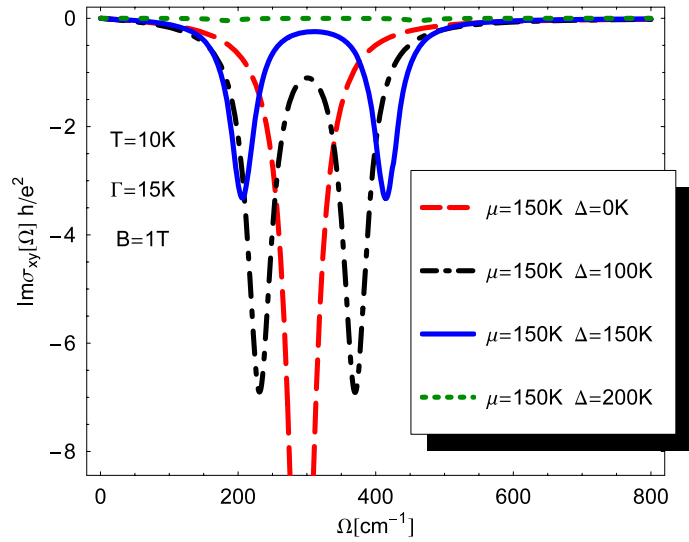
Next we consider the absorptive part of the transverse Hall conductivity. The calculations are based on equation (10). In figure 8 we show results for  $\text{Im} \sigma_{xy}(\Omega)$  in units of  $e^2/h$  as a function of  $\Omega$  in  $\text{cm}^{-1}$ . In all cases considered, temperature  $T = 10$  K, impurity scattering rate  $\Gamma = 15$  K and excitonic gap  $\Delta = 0$ . The magnetic field  $B = 1$  T and three values of chemical potential are considered. The long dashed (red) curve is for  $\mu = 50$  K below the energy of the  $n = 1$  Landau level, dash-dotted (black)  $\mu = 510$  K between  $n = 1$  and 2 and  $\mu = 660$  K between  $n = 2$  and 3 as in figure 1. We note a single peak in the long dashed (red) curve at  $M_1$ , in contrast to two in the other two curves. For the dash-dotted (black) curve the peaks are at  $M_2 - M_1$  and  $M_1 + M_2$ , respectively, and for the solid (blue) curve they are at  $M_3 - M_2$  and  $M_2 + M_3$ . These features can be easily understood from equation (10) for  $\sigma_{xy}(\Omega)$  when we take its imaginary part in the limit  $\Delta = 0$ ,  $\Gamma \rightarrow 0$  which is for  $\Omega > 0$ ,  $\mu > 0$

$$\text{Im} \sigma_{xy}(\Omega) = -\frac{e^2}{h} M_1^2 \frac{\pi}{2} \sum_{n=0}^{\infty} [n_F(M_n) - n_F(M_{n+1}) - n_F(-M_{n+1}) + n_F(-M_n)] \quad (34)$$

$$\times \left[ \frac{\delta(\Omega - M_{n+1} + M_n)}{M_{n+1} - M_n} + \frac{\delta(\Omega - M_{n+1} - M_n)}{M_{n+1} + M_n} \right]. \quad (35)$$



**Figure 8.** The imaginary part of the Hall conductivity,  $\text{Im} \sigma_{xy}(\Omega)$  in units of  $e^2/h$  as a function of frequency  $\Omega$  in  $\text{cm}^{-1}$ . The three cases are for  $\mu = 50$  K (long dashed),  $\mu = 510$  K (dash-dotted) and  $\mu = 660$  K (solid). The other parameters are  $B = 1$  T,  $T = 10$  K and  $\Gamma = 15$  K.



**Figure 9.** The imaginary part of the Hall conductivity  $\text{Im} \sigma_{xy}(\Omega)$  in units of  $e^2/h$  versus frequency  $\Omega$  in  $\text{cm}^{-1}$  for temperature  $T = 10$  K,  $\Gamma = 15$  K,  $B = 1$  T and chemical potential  $\mu = 150$  K for four values of the excitonic gap  $\Delta$ . Long dashed,  $\Delta = 0$  K; dash-dotted,  $\Delta = 100$  K; solid,  $\Delta = 150$  K; short dashed,  $\Delta = 200$  K.

Taking the limit of zero temperature,  $T = 0$ , in equation (34), only the first two thermal factors in the square bracket survive. For  $\mu \in ]M_0, M_1[$  we get

$$\text{Im} \sigma_{xy}(\Omega) = -\frac{e^2}{h} M_1^2 \pi \frac{\delta(\Omega - M_1)}{M_1}, \quad (36)$$

while for  $\mu \in ]M_N, M_{N+1}[$  with  $N > 0$  we get

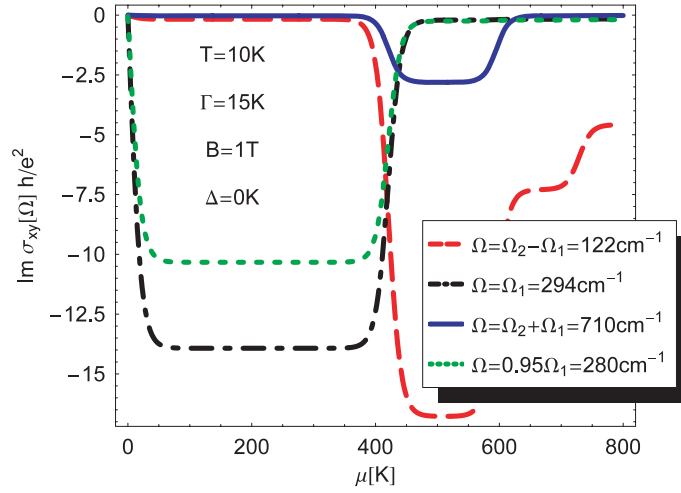
$$\text{Im } \sigma_{xy}(\Omega) = -\frac{e^2}{h} M_1^2 \frac{\pi}{2} \left[ \frac{\delta(\Omega - M_{N+1} - M_N)}{M_{N+1} + M_N} + \frac{\delta(\Omega - M_{N+1} + M_N)}{M_{N+1} - M_N} \right]. \quad (37)$$

Thus for  $\mu \in ]M_0, M_1[$  the transverse Hall conductivity exhibits a single peak at  $\Omega = M_1$  with weight  $2/M_1$ , in units of  $(e^2/h)(\pi/2)M_1^2$ , and for  $\mu \in ]M_N, M_{N+1}[$  with  $N > 0$  it has two peaks at  $M_{N+1} + M_N$  and  $M_{N+1} - M_N$  with weight  $1/(M_{N+1} + M_N)$  and  $1/(M_{N+1} - M_N)$ , respectively. Thus the case  $\mu \in ]M_0, M_1[$  is different from all others. This distinguishes Dirac from a classical Landau level quantization, for which all cases would have two peaks. Also the area under the peaks in  $\text{Im } \sigma_{xy}(\Omega)$  is  $\sim \sqrt{B}$  in analogy to what we found for  $\text{Re } \sigma_{xx}(\Omega)$ .

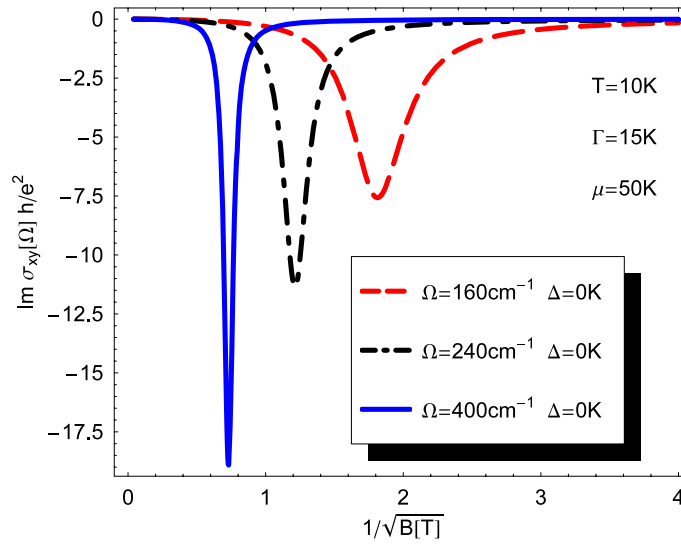
Next we consider the effect of a finite excitonic gap on the imaginary part of the Hall conductivity. In this case, even for  $\Gamma \rightarrow 0$ ,  $\Omega > 0$ ,  $\mu > 0$ , equation (10) is slightly more complicated than (34) because of the additional factors  $1 \pm \Delta^2/M_n M_{n+1}$ , which are only equal to unity when  $\Delta = 0$ . Nevertheless, to understand physically the results given in figure 9 this extra complication is not needed and we can use equation (34) as a guide. What is most important is to look at the thermal factors. We will consider only the case when the chemical potential falls between  $n = 0$  and 1 Landau levels in energy. The relevant thermal factor is  $n_F(M_0) - n_F(M_1)$ . All four curves in figure 9 have  $\mu = 150$  K. The long dashed (red) curve is for  $\Delta = 0$  K and is included for reference. It shows a single peak (in  $-\text{Im } \sigma_{xy}(\Omega)$ ) at  $294 \text{ cm}^{-1}$  as we know from our previous discussion. For the dash-dotted (black) curve  $\Delta = 100$  K, which falls below the value of chemical potential (see figure 2(b)). In this case the thermal factors in equation (34) at  $T = 0$  are unity and the delta functions correspond to  $\Omega = \sqrt{\Delta^2 + \Omega_1^2} - \Delta$  and  $\Omega = \sqrt{\Delta^2 + \Omega_1^2} + \Delta$ . Thus, the peak in the long dashed curve has split into two. Further, such a peak has approximately, but not exactly, the same optical spectral weight as that under the single peak of the long dashed line. This arises because of the weighting factors  $1 \pm \Delta^2/M_n M_{n+1}$  of equation (10) not shown explicitly in equation (34). For  $\Delta = \mu$  (solid (blue) curve) the two peaks remain at  $\Omega = \sqrt{\Delta^2 + \Omega_1^2} \pm \Delta$  (note the small shift which corresponds to the slightly different values of the gap between the solid and the short-long dashed curve). However, the thermal factor  $n_F(\Delta) - n_F(\Delta + \sqrt{\Delta^2 + \Omega_1^2})$  now equals  $1/2$  rather than 1 for the two previous cases, so this feature on its own reduces the optical spectral weight of these peaks by half. Finally, for the short dashed (green) curve the gap  $\Delta$  is larger than is the chemical potential and the thermal factor  $n_F(M_0) - n_F(M_1)$  is zero, so that no peak is seen.

In figure 10 we show results for the change in  $\text{Im } \sigma_{xy}(\Omega)$  in units of  $e^2/h$  as a function of  $\mu$  in kelvin at fixed optical frequency. This figure is the analogue of figure 5. The same parameters  $T = 10$  K,  $\Gamma = 15$  K,  $B = 1$  T and  $\Delta = 0$  are chosen as well as  $\Omega$ , namely, long dashed (red) curve,  $\Omega = \Omega_2 - \Omega_1 = 122 \text{ cm}^{-1}$ . The absorptive Hall conductivity for this curve is near zero until the first Landau level energy 420 K is crossed, where it drops below  $-17e^2/h$ , after which it shows a plateau till the next level is crossed at  $\mu = 595$  K, where the second step is seen, etc. The dash-dotted (black) curve is for  $\Omega = \Omega_1 = 294 \text{ cm}^{-1}$ . In this case the first plateau is at  $\sim -14e^2/h$  till the energy of the  $n = 1$  Landau level is crossed, in which case it drops to near zero value. The short dashed (green) curve is for  $\Omega = 0.95\Omega_1 = 280 \text{ cm}^{-1}$  and follows the dash-dotted (black) curve, except the plateau is at  $\sim -10e^2/h$  as we expect. Finally, the solid (blue) curve is for  $\Omega = \Omega_1 + \Omega_2 = 710 \text{ cm}^{-1}$ . It starts at zero till  $\mu$  crosses 420 K, at which point it shows a step down, remains nearly constant and finally steps back to near zero value at  $\mu = 595$  K as expected from consideration of figure 8.

The final figure 11 shows results for the variation of  $\text{Im } \sigma_{xy}(\Omega)$  in units of  $e^2/h$  versus the inverse square root of the magnetic field,  $1/\sqrt{B(T)}$ . Here temperature  $T = 10$  K, impurity



**Figure 10.** The imaginary part of the Hall conductivity  $\text{Im } \sigma_{xy}(\Omega)$  in units of  $e^2/h$  as a function of chemical potential  $\mu$  in K. Four frequencies are considered: long dashed,  $\Omega = \Omega_2 - \Omega_1 = 122\text{ cm}^{-1}$ ; dash-dotted,  $\Omega = \Omega_1 = 294\text{ cm}^{-1}$ ; solid,  $\Omega = \Omega_2 + \Omega_1 = 710\text{ cm}^{-1}$ ; short dashed,  $\Omega = 0.95\Omega_1 = 280\text{ cm}^{-1}$ . The other parameters are  $B = 1\text{ T}$ ,  $T = 10\text{ K}$  and  $\Gamma = 15\text{ K}$ .



**Figure 11.** The imaginary part of the Hall conductivity  $\text{Im } \sigma_{xy}(\Omega)$  in units of  $e^2/h$  as a function of the inverse of the square root of the magnetic field,  $1/\sqrt{B}$ , with  $B$  in T. Three frequencies are considered: long dashed,  $160\text{ cm}^{-1}$ ; dash-dotted,  $240\text{ cm}^{-1}$ ; solid,  $400\text{ cm}^{-1}$ . The other parameters are  $T = 10\text{ K}$ ,  $\Gamma = 15\text{ K}$  and  $\mu = 50\text{ K}$ .

scattering rate  $\Gamma = 15\text{ K}$  and chemical potential  $\mu = 50\text{ K}$  with excitonic gap  $\Delta = 0$ . Three optical frequencies are chosen: long dashed (red) curve,  $\Omega = 160\text{ cm}^{-1}$ ; dash-dotted (black) curve,  $\Omega = 240\text{ cm}^{-1}$ ; solid (blue) curve,  $\Omega = 400\text{ cm}^{-1}$ . These curves are similar to those of figure 6 for the longitudinal case, but here exhibit only a single peak as is expected from the curves shown in figure 8.

#### 4. Discussion

In this paper we have extended the calculation of the ac conductivity of graphene presented in [8] in several ways. One emphasis has been on presenting results for different values of the chemical potential,  $\mu$ . In a field-effect device,  $\mu$  can be varied within the range of a few thousand kelvin by changing the gate voltage [1–3, 17]. This implies that for fields of the order of 1 T,  $\mu$  can be made to sweep over several Landau levels. Recently [18], infrared spectroscopy has been successfully applied to study FET devices based on poly(3-hexylthiophene). We find a rich pattern of behaviour for the  $\Omega$  dependence of the real (absorptive) part of the diagonal conductivity,  $\text{Re } \sigma_{xx}(\Omega, T)$ , as a function of  $\Omega$  in a fixed external magnetic field oriented perpendicular to the graphene sheet. Peaks are seen in  $\sigma_{xx}(\Omega, T)$  corresponding to the possible transition between Landau levels from the hole to particle band (interband) or within a given cone (intraband). For  $\mu$  between the  $n = 0$  and 1 levels there are lines at  $\Omega = M_1, M_1 + M_2, M_2 + M_3$  (interband) etc, where  $M_i$  is the position in energy of the  $i$ th Landau level. There is no peak below  $M_1$ . The relative optical spectral weight between the various levels decays approximately as  $1/\Omega$ , with  $\Omega$  evaluated at the centre of each peak. When the chemical potential falls between the energy of the  $n = 1$  and 2 levels the peak at  $\Omega = M_1$  fades into the background and a new line appears at the lower frequency  $M_2 - M_1$  (intraband). In addition, the intensity of the line at  $M_1 + M_2$  drops to half its value, while all other lines remain the same. As the chemical potential crosses higher and higher Landau levels, say falls between  $M_N$  and  $M_{N+1}$ , the low energy peak has shifted to  $M_{N+1} - M_N$  (intraband), after which all peaks previously seen have disappeared into the background except for the one at  $M_{N+1} + M_N$ , which has half its previous intensity. Again, all peaks above this energy remain unaltered. The peculiar behaviour of the peak at  $\Omega = M_1$ , which is either present with full intensity or completely absent, is the hallmark of Dirac as opposed to Schrödinger behaviour. This peak can be classified either as inter or intra when the excitonic gap  $\Delta = 0$ , an ambiguity that is lifted for  $\Delta \neq 0$ . For Landau levels based on the Schrödinger equation the interband lines are evenly spaced while the intraband line is fixed in energy, and all interband lines first halve their intensity before disappearing entirely as  $\mu$  is increased.

We have found that the transition from one configuration of absorption lines to another occurs for a small change in chemical potential near a given Landau level energy with the scale for the incremental change in  $\mu$  set by temperature and/or level broadening  $\Gamma$ . Away from these special values of  $\mu$  the curves do not change significantly.

In anticipation of experiments [32] we also provide scans of the behaviour of  $\text{Re } \sigma_{xx}(\Omega, T)$  at fixed  $\Omega$  as a function of chemical potential or of magnetic field. The pattern of behaviour found is traced back to that just described for  $\text{Re } \sigma_{xx}(\Omega, T)$  versus  $\Omega$  at different values of field and chemical potential. Parallel results for the absorptive part of the transverse Hall conductivity are also presented.

The possibility that an excitonic gap may open in graphene under high magnetic field has been considered by many authors [19–21, 24] and may even have been observed in recent experiments [23]. We have considered its effect on the absorption peak seen in both diagonal and Hall conductivity. Our specific predictions are that a given peak can split into two, can disappear entirely or can simply shift to higher energy without splitting depending on the value of the chemical potential. In some circumstances the optical spectral weights under the split peak can differ from each other by a factor of order three.

Many of the results shown in the figures were obtained on the basis of general formulae for the conductivity, but it was found that simplified versions which can be more easily used to

interpret experimental results are surprisingly accurate in the cases considered. They involve sums over Landau level indices of Lorentzian forms. Recently Li *et al* [22] and Sadowski *et al* [29] used related forms to analyse their results.

Early magnetorefectance data [33] in magnetic fields in the range 1–10 T carried out in graphite assigned lines to the  $H$ -point and some of these lines were found to follow a square root of  $B$  law as expected for graphene. Recent data of [22] at higher fields up to 20 T, however, found a conventional linear in  $B$  behaviour. Very recent infrared transmission data [29] in ultrathin epitaxial graphite [30, 31] in fields up to 4 T do show  $\sqrt{B}$  law and these authors took this to be evidence that their carbon sheets are sufficiently decoupled to behave like graphene. However, this interpretation also requires that they assume that various sheets carry different charges, i.e. have different values of chemical potential. Nevertheless, the main lines seen in these data were assigned to the first three Dirac interband transitions and the (0, 1) intraband transition on the basis of their position in energy. They were also seen to vary in optical spectral weight as the square root of the magnetic field, in good agreement with the findings in this work. Many other detailed predictions made in this paper have yet to be verified and should help in firmly establishing the special characteristic of quasiparticles in graphene.

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### Appendix. Calculation of $\sigma_{\pm}(\Omega)$ in magneto-optical Lorentzian model

We consider the complex conductivity  $\sigma_{\pm}(\Omega) = \sigma_{xx}(\Omega) \pm i\sigma_{xy}(\Omega)$ . Substituting the spectral function (3) in equation (2) after evaluating tr and integrating over momentum  $\mathbf{k}$  (see appendix A of [8]) one obtains

$$\sigma_{\pm}(\Omega) = \frac{e^2 N_{\text{F}} v_{\text{F}}^2 |eB|}{4\pi c \Omega i} \int_{-\infty}^{\infty} d\omega d\omega' \frac{n_{\text{F}}(\omega') - n_{\text{F}}(\omega)}{\omega - \omega' - \Omega - i0} [\psi_1(\omega, \omega') \mp \text{sgn}(eB)\psi_2(\omega, \omega')], \quad (\text{A.1})$$

where the functions  $\psi_1(\omega, \omega')$  and  $\psi_2(\omega, \omega')$  are

$$\psi_{1,2}(\omega, \omega') = \sum_{n,m=0}^{\infty} (-1)^{n+m+1} (\delta_{n,m-1} \pm \delta_{m,n-1}) \psi_{n,m}(\omega, \omega') \quad (\text{A.2})$$

with

$$\begin{aligned} \psi_{n,m}(\omega, \omega') &= \left(1 - \frac{\Delta^2}{M_n M_m}\right) (A_n(\omega)A_m(\omega') + B_n(\omega)B_m(\omega')) \\ &+ \left(1 + \frac{\Delta^2}{M_n M_m}\right) (A_n(\omega)B_m(\omega') + B_n(\omega)A_m(\omega')), \end{aligned} \quad (\text{A.3})$$

and

$$A_n(\omega) = \frac{\Gamma_n}{\pi[(\omega - M_n)^2 + \Gamma_n^2]}, \quad B_n(\omega) = \frac{\Gamma_n}{\pi[(\omega + M_n)^2 + \Gamma_n^2]}. \quad (\text{A.4})$$

One can easily check that

$$\psi_1(\omega, \omega') = \psi_1(\omega', \omega), \quad \psi_2(\omega, \omega') = -\psi_2(\omega', \omega). \quad (\text{A.5})$$

Using this symmetry one can rewrite equation (A.1) in the form

$$\begin{aligned} \sigma_{\pm}(\Omega) = & -\frac{e^2 N_{\text{F}} v_{\text{F}}^2 |eB|}{4\pi c \Omega i} \int_{-\infty}^{\infty} d\omega n_{\text{F}}(\omega) \\ & \times \int_{-\infty}^{\infty} d\omega' \left\{ \left[ \frac{1}{\omega - \omega' + \Omega + i0} + \frac{1}{\omega - \omega' - \Omega - i0} \right] \psi_1(\omega, \omega') \right. \\ & \left. \pm \operatorname{sgn}(eB) \left[ \frac{1}{\omega - \omega' + \Omega + i0} - \frac{1}{\omega - \omega' - \Omega - i0} \right] \psi_2(\omega, \omega') \right\}. \end{aligned} \quad (\text{A.6})$$

Now using equation (A.2) we obtain

$$\begin{aligned} \sigma_{\pm}(\Omega) = & -\frac{e^2 N_{\text{F}} v_{\text{F}}^2 |eB|}{4\pi c \Omega i} \sum_{n,m=0}^{\infty} (-1)^{n+m+1} \int_{-\infty}^{\infty} d\omega n_{\text{F}}(\omega) \int_{-\infty}^{\infty} d\omega' \psi_{n,m}(\omega, \omega') \\ & \times \left\{ \left[ \frac{1}{\omega - \omega' + \Omega + i0} + \frac{1}{\omega - \omega' - \Omega - i0} \right] (\delta_{n,m-1} + \delta_{m,n-1}) \right. \\ & \left. \pm \operatorname{sgn}(eB) \left[ \frac{1}{\omega - \omega' + \Omega + i0} - \frac{1}{\omega - \omega' - \Omega - i0} \right] (\delta_{n,m-1} - \delta_{m,n-1}) \right\}. \end{aligned} \quad (\text{A.7})$$

For a typical integral over  $\omega'$  we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\omega' \psi_{n,m}(\omega, \omega')}{\omega - \omega' + \Omega + i0} = & \left( 1 - \frac{\Delta^2}{M_n M_m} \right) \left( \frac{A_n(\omega)}{\omega - M_m + \Omega + i\Gamma_m} + \frac{B_n(\omega)}{\omega + M_m + \Omega + i\Gamma_m} \right) \\ & + \left( 1 + \frac{\Delta^2}{M_n M_m} \right) \left( \frac{B_n(\omega)}{\omega - M_m + \Omega + i\Gamma_m} + \frac{A_n(\omega)}{\omega + M_m + \Omega + i\Gamma_m} \right). \end{aligned} \quad (\text{A.8})$$

If the thermal factor  $n_{\text{F}}$  is absent, further integration over  $\omega$  would give an exact result:

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega d\omega' \frac{\psi_{n,m}(\omega, \omega')}{\omega - \omega' + \Omega + i0} = & \left( 1 - \frac{\Delta^2}{M_n M_m} \right) \left( \frac{1}{M_n - M_m + \Omega + i(\Gamma_n + \Gamma_m)} \right. \\ & \left. + \frac{1}{-M_n + M_m + \Omega + i(\Gamma_n + \Gamma_m)} \right) + \left( 1 + \frac{\Delta^2}{M_n M_m} \right) \\ & \times \left( \frac{1}{-M_n - M_m + \Omega + i(\Gamma_n + \Gamma_m)} + \frac{1}{M_n + M_m + \Omega + i(\Gamma_n + \Gamma_m)} \right). \end{aligned} \quad (\text{A.9})$$

If we integrate with a smooth function  $n_{\text{F}}(\omega)$  we can approximately write

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega n_{\text{F}}(\omega) \int_{-\infty}^{\infty} \frac{d\omega' \psi_{n,m}(\omega, \omega')}{\omega - \omega' + \Omega + i0} \simeq & \left( 1 - \frac{\Delta^2}{M_n M_m} \right) \left( \frac{n_{\text{F}}(M_n)}{M_n - M_m + \Omega + i(\Gamma_n + \Gamma_m)} \right. \\ & \left. + \frac{n_{\text{F}}(-M_n)}{-M_n + M_m + \Omega + i(\Gamma_n + \Gamma_m)} \right) + \left( 1 + \frac{\Delta^2}{M_n M_m} \right) \\ & \times \left( \frac{n_{\text{F}}(-M_n)}{-M_n - M_m + \Omega + i(\Gamma_n + \Gamma_m)} + \frac{n_{\text{F}}(M_n)}{M_n + M_m + \Omega + i(\Gamma_n + \Gamma_m)} \right). \end{aligned} \quad (\text{A.10})$$



Similarly, taking the complex conjugate and changing  $\Omega \rightarrow -\Omega$ , we have

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega n_F(\omega) \int_{-\infty}^{\infty} \frac{d\omega' \psi_{n,m}(\omega, \omega')}{\omega - \omega' - \Omega - i0} \simeq & \left(1 - \frac{\Delta^2}{M_n M_m}\right) \left(\frac{n_F(M_n)}{M_n - M_m - \Omega - i(\Gamma_n + \Gamma_m)}\right. \\ & \left. + \frac{n_F(-M_n)}{-M_n + M_m - \Omega - i(\Gamma_n + \Gamma_m)}\right) + \left(1 + \frac{\Delta^2}{M_n M_m}\right) \\ & \times \left(\frac{n_F(-M_n)}{-M_n - M_m - \Omega - i(\Gamma_n + \Gamma_m)} + \frac{n_F(M_n)}{M_n + M_m - \Omega - i(\Gamma_n + \Gamma_m)}\right). \end{aligned} \quad (\text{A.11})$$

Hence we arrive at

$$\begin{aligned} \sigma_{\pm}(\Omega) = & -\frac{e^2 N_f v_F^2 |eB|}{4\pi c \Omega i} \sum_{n=0}^{\infty} \left\{ \left(1 - \frac{\Delta^2}{M_n M_{n+1}}\right) [[n_F(M_n) - n_F(M_{n+1})] \right. \\ & + [n_F(-M_{n+1}) - n_F(-M_n)]] \\ & \times \left[ \frac{1}{M_n - M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} + \frac{1}{M_n - M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right] \\ & - \left(1 + \frac{\Delta^2}{M_n M_{n+1}}\right) [[n_F(-M_{n+1}) - n_F(M_n)] + [n_F(-M_n) - n_F(M_{n+1})]] \\ & \times \left[ \frac{1}{M_n + M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} + \frac{1}{M_n + M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} \right] \\ & \pm \operatorname{sgn}(eB) \left[ \left(1 - \frac{\Delta^2}{M_n M_{n+1}}\right) [[n_F(M_n) - n_F(M_{n+1})] \right. \\ & - [n_F(-M_{n+1}) - n_F(-M_n)]] \\ & \times \left[ \frac{1}{M_n - M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} - \frac{1}{M_n - M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right] \\ & + \left(1 + \frac{\Delta^2}{M_n M_{n+1}}\right) [[n_F(-M_{n+1}) - n_F(M_n)] - [n_F(-M_n) - n_F(M_{n+1})]] \\ & \times \left[ \frac{1}{M_n + M_{n+1} - \Omega - i(\Gamma_n + \Gamma_{n+1})} \right. \\ & \left. \left. - \frac{1}{M_n + M_{n+1} + \Omega + i(\Gamma_n + \Gamma_{n+1})} \right] \right\}, \end{aligned} \quad (\text{A.12})$$

where only the sum over  $n$  remains.

We verified that  $\operatorname{Re} \sigma_{xx}(\Omega)$  computed for  $\Gamma_n(\omega) = \text{const}$  from the more approximate equation (A.12) agrees quantitatively with the results obtained from a full equation (7). This agreement is best for the resonance peaks and only small deviations are seen for  $\Omega \sim 0$ . Nevertheless, equation (A.12) has a few drawbacks due to approximations made in equations (A.10) and (A.11).

In particular, the Drude form cannot be recovered in the  $B \rightarrow 0$  limit, while it can be obtained [8, 16] from an exact representation (7). Moreover, the imaginary parts of the diagonal conductivity,  $\operatorname{Im} \sigma_{xx}(\Omega)$ , and the Hall conductivity,  $\operatorname{Im} \sigma_{xy}(\Omega)$ , are divergent in the limit  $\Omega \rightarrow 0$  and do not satisfy Kramers–Kronig relations with the corresponding real parts found from equation (A.12). To correct these problems we move the term  $1/\Omega$  under the sum, replacing it by its value at the pole of the corresponding denominator in equation (A.12), and arrive at equations (9) and (10).

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